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p-ADIC GAMMA FUNCTIONS AND DWORK COHOMOLOGY

BY

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ABSTRACT. The relations of Gross and Koblitz between gauss sums and the p-adic gamma function is reexamined from the point of view of Dwork's formulation of p-adic cohomology. Some higher dimensional generalizations are proposed.

Introduction. Let $q = p^f$, $p \neq 2$, $d|(q-1)$. Let θ be a nontrivial additive character of $\mathbf{F}_q$ and let χ be a character of $\mathbf{F}_q^*/\mathbf{F}_q^{*d}$. The gauss sum

$$(\theta, \chi) = \sum_{x \in \mathbf{F}_q^*} \theta(x)\chi(x)$$

has recently been described by B. Gross and N. Koblitz (equation (5.13) below) as a product of values $\Gamma_p(i/d)$ assumed at certain rational numbers by the p-adic gamma function studied by Morita [5].

Some time ago [1, §4b] Dwork had associated certain constant matrix coefficients μ_j with sums of the form $\Sigma_{x \in \mathbf{F}_q^*} \theta(x^d)$ and noted their relation to gauss sums. In a subsequent article [2, §6k] he made explicit the relation between the gauss sum (θ, χ) and the μ_j (cf. (4.4) below).

In the present article we give the relation ((5.12)) between the μ_j and the values of the Γ_p function and this shows the consistency of the two formulas. In the process we shall obtain some known properties of the Γ_p-function. In §7 we give the cohomologic interpretation of the beta function and in §8 we give the multiplication formula for Γ_p. In that section we mention some previously unpublished identities involving gauss sums and give the reader the task of verifying them on the basis of the Gross-Koblitz formula.

It is understood that when cohomology depends analytically upon a parameter then the Frobenius operation will also "vary" analytically with the parameter (cf. N. Katz, Séminaire Bourbaki (1971/72), exposé 409, Lecture Notes in Math., vol. 317, Springer, New York, pp. 167–200). The present article gives a one-dimensional example of the "principle" that when the cohomology is parametrized say rationally by a *character* then the Frobenius operation will vary *continuously* with the character.

We briefly indicate a possible two dimensional example. The Sonine cohomology for $J_0(x)$ was explained by Dwork in [3, §1]. The corresponding cohomology for $J_\alpha(x)$ with $\alpha \in \mathbf{Z}_p \cap \mathbf{Q}$ can be similarly formulated. It should depend essentially

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only upon α mod $\mathbf{Z}$ (cf. equation (6.3) below). Denoting this space by $\mathbf{W}_{x,\alpha}$ (instead of just $\mathbf{W}_x$ as in [3, §2]) the inverse Frobenius should map $\mathbf{W}_{x,\alpha}$ into $\mathbf{W}_{x^p,\alpha^{(1)}}$ where $p\alpha^{(1)} \equiv \alpha$ mod $\mathbf{Z}$. The matrix of this map relative to the obvious basis should vary analytically with x and continuously with α.

Similar examples could be formulated in terms of the hypergeometric series $F(\alpha, \beta, \gamma, x)$ using the Euler cohomology. Higher dimensional examples are suggested by the work of S. Sperber (Duke Math. J. **44** (1977)) based on the integral formulas of Erdelyi.

Similar problems may be posed for systems of ordinary linear differential equations. Here the exponents at the singular points should play the role of characters.

1. Morita gamma function. We recall some properties of Γ_p. For $n \in \mathbf{N}$ we set

$$\Gamma_p(n) = (-1)^n \prod_{\substack{i=1 \\ p\nmid i}}^{n-1} i \tag{1.1}$$

It is known [5] that this extends to a continuous function on $\mathbf{Z}_p$, and Γ_p is defined to be this extension. Hence by continuity one has the functional equation

$$\begin{aligned} \Gamma_p(x+1)/\Gamma_p(x) &= -1, \qquad x \in p\mathbf{Z}_p, \\ &= -x \qquad x \notin p\mathbf{Z}_p, \end{aligned} \tag{1.2}$$

and also

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{l(x)} \tag{1.3}$$

where $l(x)$ is the representative of x mod p in $\{1, 2, \ldots, p\}$. Equation (1.3) will be explained in terms of duality (cf. (3.9), (6.12)).

2. Review of Dwork's notation [1, §4b]. Let $\pi \in \mathbf{C}_p$, $\pi^{p-1} = -p$; let γ_- be the projection of formal Laurent series in X (with coefficients in $\mathbf{C}_p$) onto $X^{-1}\mathbf{C}_p[[X^{-1}]]$ defined by linearity and

$$\begin{aligned} \gamma_- X^\nu &= X^\nu \quad \text{if } \nu < 0, \\ &= 0 \quad \text{if } \nu > 0. \end{aligned}$$

Let

$$E = X\frac{d}{dX}, \qquad D_1 = E - \pi d X^d,$$

$$D = E + \pi d X^d, \qquad D^* = \gamma_- \circ D,$$

and let $\mathbf{K}$ be the kernel of D^* in $X^{-1}\mathbf{C}_p[[X^{-1}]]$. Then $\xi_1^*, \ldots, \xi_{d-1}^*$ is a basis of $\mathbf{K}$ where

$$\xi_j^* = X^{-j} \sum_{r=0}^{\infty} \left(\frac{j}{d}\right)_r \frac{1}{(\pi X^d)^r} \tag{2.1}$$

and $(b)_r = \Gamma(b+r)/\Gamma(b)$, Γ being the ordinary Γ-function. Let $F(X) = \exp \pi(X^p - X)$ and we write

$$F(X) = \sum_{s=0}^{\infty} A_s X^s, \tag{2.2}$$

where all the $A_s \in \mathbf{Q}_p(\pi)$ and

$$\operatorname{ord} A_s \geqslant s(p-1)/p^2. \tag{2.3}$$

We recall the operators ϕ, ψ on Laurent series defined by linearity and

$$\phi(X^\nu) = X^{\nu p},$$
$$\psi(X^\nu) = X^{\nu/p} \quad \text{if } p|\nu,$$
$$= 0 \quad p \nmid \nu.$$

We now recall the definition of the μ_j. They are the matrix coefficients of the endomorphism $\alpha^* = \gamma_- F(X^d) \circ \phi$ of $\mathbf{K}$. Explicitly

$$\alpha^* \xi_j^* = \mu_j \xi_{j'}^*, \tag{2.4}$$

where j' = minimal positive representative of pj mod d.

3. Review of duality. Let L be the ring of power series converging in disks of center 0 and radii strictly greater than 1. Under the pairing

$$\langle \xi^*, \xi \rangle = \text{constant term of } \xi^* \xi, \tag{3.1}$$

$\mathbf{K}$ is dual to XL/D_1L. Under this pairing α^* is dual to $\alpha_1 = \psi \circ F(X^d)$. Also $X, X^2, \ldots, X^{d-1}$ is basis of XL/D_1L dual to $\xi_1^*, \ldots, \xi_{d-1}^*$ and so by duality

$$\alpha_1 X^{j'} \in u_j X^j + D_1 L. \tag{3.2}$$

Applying $\alpha_1^{-1} = (1/F(X^d)) \circ \phi$ (it is the inverse of α_1 in its action on the factor space),

$$X^{j'} \in \mu_j \frac{1}{F(X^d)} X^{jp} \bmod D_1 L. \tag{3.3}$$

The desired dual relation involves L/DL. It is obtained by replacing X by $X(-1)^{1/d}$:

$$X^{j'} \in \mu_j \varepsilon_j X^{jp} F(X^d) \mod DL \tag{3.4}$$

where $\varepsilon_j = (-1)^{(pj-j')/d}$ and so

$$\psi \frac{1}{F(X^d)} X^{j'} \in \mu_j \varepsilon_j X^j \mod DL. \tag{3.5}$$

For the gauss sum formula we will need the f-fold iterate of this formula. Under the f-fold iteration of $j' \to j$, we must again obtain j'.

Now let $j_0, j_1, \ldots, j_{f-1}$ be the minimal residues mod d of $j, pj, \ldots, p^{f-1}j$. Then

$$\left(\psi \circ \frac{1}{F(X^d)}\right)^f X^j \equiv \prod_{i=0}^{f-1} (\mu_{j_i} \varepsilon_{j_i}) \cdot X^j \mod DL. \tag{3.6}$$

By definition of ξ_j^*,

$$D\xi_j^* = \pi d X^{d-j} \tag{3.7}$$

while formally

$$(F(X^d) \circ \phi) \circ D = \frac{1}{p} D \circ (F(X^d) \circ \phi); \tag{3.8}$$

applying this to ξ_j^*, we calculate the left side using first (3.7) and then (3.4), while for the right side we use (2.4) followed by (3.7). We obtain

$$p = \mu_j \mu_{d-j} \varepsilon_{d-j}. \tag{3.9}$$

4. Gauss sums in terms of the μ_j. Let $\theta_f(x) = \exp \pi(x - x^q)$. We define $(0 < j < d)$:

$$-g_f\left(\frac{j}{d}(q-1)\right) = \sum_{x^{q-1}=1} x^{-(j/d)(q-1)}\theta_f(x), \tag{4.1}$$

the sum being over the $(q-1)$ roots of unity in $\mathbf{C}_p$. For $\bar{x} \in \mathbf{F}_q$, let Teich $\bar{x}$ be the representative x of $\bar{x}$ in $\mathbf{C}_p$ which satisfies the equation $x^q - x = 0$. The mapping

$$\bar{x} \mapsto \text{Teich } \bar{x} \mapsto \theta_f(\text{Teich } \bar{x}) \tag{4.2}$$

is an additive character of $\mathbf{F}_q$ and hence (4.1) defines a gauss sum.

It is known [2, §6k] that

$$\psi^f X^j \theta_f(X^d) \equiv g_f\left(\frac{j}{d}(q-1)\right)X^j \mod DL. \tag{4.3}$$

The left side of (4.3) may be written $(\psi \circ (F(X^d))^{-1})^f X^j$ and so by (3.6)

$$g_f\left(\frac{j}{d}(q-1)\right) = \prod_{i=0}^{f-1} \mu_{j_i}\varepsilon_{j_i} \tag{4.4}$$

where $j_0, j_1, \ldots, j_{f-1}$ are the representatives mod d of $j, pj, \ldots, p^{f-1}j$. This may be referred to as Dwork's formula for gauss sums.

5. The μ_j as values of Γ_p. Let j and j' be as in equation (2.4). We write $a = j/d$, $a' = j'/d$ and in general if y is real number then $\langle y \rangle$ is defined by $y \equiv \langle y \rangle$ mod $\mathbf{Z}$, $0 < \langle y \rangle < 1$; thus $a' = \langle pa \rangle$. We write $C(a) = \mu_j$ (we make no assertion at this point concerning μ_{mj} for d replaced by md, $m \in \mathbf{Z}$).

By comparing for $t \in \mathbf{N}$, the coefficients of $1/X^{j'+td}$ on the two sides of (2.4) we obtain

$$\mu_j\left(\frac{j'}{d}\right)_t \frac{1}{\pi^t} = \sum A_s\left(\frac{j}{d}\right)_r \frac{1}{\pi^r} \tag{5.1}$$

the sum on the right being over all $r, s > 0$ such that $-ds + jp + pdr = dt + j'$, i.e. $-s + (jp - j')/d + pr = t$. Let $t_0 = (jp - j')/d$ and let $t = t_0 + pz$, $z \in \mathbf{N}$, then (5.1) becomes

$$\mu_j\left(\frac{j'}{d}\right)_{t_0+pz} \frac{1}{\pi^{t_0+pz}} = \sum_{r=z}^{\infty} A_{p(r-z)}\left(\frac{j}{d}\right)_r \frac{1}{\pi^r}. \tag{5.2}$$

With a, a', $C(a)$ as defined above, equation (5.2) takes the form

$$C(a)(a')_{t_0+pz}\frac{1}{\pi^{t_0+pz}} = \sum_{r=0}^{\infty} A_{pr}(a)_{r+z}\frac{1}{\pi^{r+z}} \tag{5.3}$$

where $t_0 = [pa] = pa - a'$. But

$$(a)_{r+z} = \frac{\Gamma(a+r+z)}{\Gamma(a)} = \frac{\Gamma(a+r+z)}{\Gamma(a+z)}\frac{\Gamma(a+z)}{\Gamma(a)} = (a+z)_r(a)_z \tag{5.4}$$

and now (5.3) becomes

$$C(a)(a')_{t_0+pz}\frac{1}{\pi^{t_0+pz}} = \frac{(a)_t}{\pi^z}\, G(a+z), \tag{5.5}$$

where

$$G(a) = \sum_{r=0}^{\infty} A_{pr}\frac{(a)_z}{\pi^r}. \tag{5.6}$$

We will show that $G(a) = \Gamma_p(pa)$ for all $a \in \mathbf{Z}_p$. For the moment we note (by (2.3)) that

$$\operatorname{ord}\left(\frac{A_{pr}}{\pi^r}\right) > r\left(\frac{p-1}{p} - \frac{1}{p-1}\right)$$

and so G is analytic on $D(0, 1+\varepsilon)$ if $p > 3$.

Putting $z = 0$ in (5.5) gives

$$C(a)(a')_{t_0}/\pi^{t_0} = G(a). \tag{5.7}$$

Taking the ratio with (5.5) gives

$$\frac{(a')_{t_0+pz}}{(a')_{t_0}}\frac{1}{\pi^{pz}} = \frac{(a)_z}{\pi^z}\frac{G(a+z)}{G(a)}. \tag{5.8}$$

Using equation (5.4),

$$(a')_{t_0+pz} = (a')_{t_0}(a'+t_0)_{pz} = (a')_{t_0}(pa)_{pz}$$

and so from (5.8)

$$\frac{G(a+z)}{G(a)} = \frac{(pa)_{pz}}{(a)_z}\frac{1}{(-p)^z}. \tag{5.9}$$

We write

$$\frac{(pa)_{pz}}{(a)_z} = \frac{pa(pa+1)\cdots(pa+pz-1)}{a(a+1)\cdots(a+z-1)}$$
$$= \frac{(pa+1)(pa+2)\cdots(pa+pz)}{(a+1)(a+2)\cdots(a+z)}$$

taking the limit as $a \to 0$ (with z fixed in $\mathbf{N}$, $z > 0$) and comparing with (5.9), we obtain

$$G(z) = \frac{(pz)!}{z!}\frac{1}{(-p)^z} = \Gamma_p(pz). \tag{5.10}$$

This shows that $z \mapsto \Gamma_p(pz)$ extends to an analytic function $D(0, 1+\varepsilon)$ for some $\varepsilon > 0$ (and can clearly be used to give a new demonstration of the results of §1). From (5.7) we now have

$$C(a)(a')_{t_0}\frac{1}{\pi^{t_0}} = \Gamma_p(pa). \tag{5.11}$$

But

$$(a')_{t_0} = a'(a'+1)\cdots(a'+t_0-1)$$

and none of the factors is divisible by p (by definition of t). Hence by the functional equation of Γ_p,

$$(a')_{t_0} = \frac{\Gamma_p(a' + t_0)}{\Gamma_p(a')}(-1)^{t_0} = \frac{\Gamma_p(pa)}{\Gamma_p(a')}(-1)^{t_0}.$$

Using this in (5.11) gives

$$C(a)(-1)^{pa-\langle pa\rangle} = \pi^{pa-\langle pa\rangle}\Gamma_p(\langle pa\rangle). \tag{5.12}$$

This gives the relation between the μ_j and the values of Γ_p. The Gross-Koblitz formula is obtained by substituting equation (5.12) in (4.4); explicitly let

$$\frac{j}{d}(p^f - 1) = c_0 + c_1 p + \cdots + c_{f-1}p^{f-1},$$
$$0 < c_i < p, \quad i = 0, 1, \ldots, f-1,$$

then

$$g_f\left(\frac{j}{d}(p^f - 1)\right) = \pi^{c_0 + c_1 + \cdots + c_{f-1}} \prod_{i=0}^{f-1} \Gamma_p\left(\left\langle p^i \frac{j}{d}\right\rangle\right). \tag{5.13}$$

A further consequence of (5.12) is that the duality formula (3.9) may be identified with equation (1.3). Finally we observe that equations (5.10) shows that the right side of (5.6) gives an interpolation series over the negative integers for $\Gamma_p(pa)$ and the coefficients are related to the coefficients of $F(x)$ (cf. the article by D. Barsky, *On Morita's p-adic* Γ *function*, Groupe d'Étude d'Analyse Ultramétrique, 1977/78, IHP).

6. Washnitzer-Monsky cohomology. In classical analysis one represents the gamma function by the second eulerian integral,

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}, \qquad \text{Re } z > 0. \tag{6.1}$$

To obtain analagous results for Γ_p we follow the spirit of Dwork's explanation [3, §1] of a cohomology for Bessel functions. As in §3, let L denote the space of power series with coefficients in $\mathbf{C}_p$ which converge in disks of radii greater than unity. We take the space of 0-forms, Ω^0, to consist of finite sums of terms $X^a\xi$ $\exp \pi X$, where $a \in \mathbf{Z}_p \cap \mathbf{Q}$ and $\xi \in L$ (thus a typical 0-form would be $(\sum_{i=1}^n X^{a_i}\xi_i)$ $\exp \pi X$ where $a_1, \ldots, a_n \in \mathbf{Z}_p \cap \mathbf{Q}$ and each $\xi_i \in L$). We define Ω^1 the space of 1-forms to be $\Omega^0 dX/X$. Our classes are elements of the factor space, $\Omega^1/d\Omega^0$. Let ω_a denote the 1-form

$$\omega_a = X^a \exp \pi X \frac{dX}{X}. \tag{6.2}$$

Since $a\omega_a + \pi\omega_{a+1} = d(X^a \exp \pi X)$ we conclude that if $a \equiv b \bmod \mathbf{Z}$ then

$$(-\pi)^a \omega_a/\Gamma(a) \equiv (-\pi)^b \omega_b/\Gamma(b) \mod d\Omega^0 \tag{6.3}$$

(note that Γ, not Γ_p, appears in this relation).

We define the operator ϕ on Ω^0 by

$$X^a\xi \exp \pi X \to X^{pa}\xi(X^p)\exp \pi(X^p - X) \cdot \exp \pi X \tag{6.4}$$

and we extend this to 1-forms $\eta dX/X$, $\eta \in \Omega^0$, by

$$\phi(\eta dX/X) = \phi(\eta)\frac{dX}{X}. \tag{6.5}$$

Note that ϕ is a transformation of Ω^0 into itself since $\exp \pi(X^p - X) \in L$. Let $\eta \in \Omega^0$, then $d\eta = X(d\eta/dX) \cdot (dX/X)$ and it follows from (6.5) that

$$\phi(d\eta) = \frac{1}{p} X \frac{d}{dX}(\phi\eta) \cdot \frac{dX}{X} = \frac{1}{p}\, d(\phi\eta) \tag{6.6}$$

and thus ϕ acts on $\Omega^1/d\Omega^0$. In particular, equation (3.5) is equivalent to

$$\phi(\omega_a) \equiv \frac{1}{\pi^{t_0}\Gamma_p(a')}\, \omega_{a'} \mod d\Omega^0 \tag{6.7}$$

where $0 < a < 1$, $a \in \mathbf{Q} \cap \mathbf{Z}_p$, $a' = \langle pa \rangle$, $t_0 = pa - a' \in \mathbf{Z}$.

7. Beta function. The analogue of the first eulerian integral,

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad (\operatorname{Re} x > 0, \operatorname{Re} y > 0) \tag{7.1}$$

is given by Davenport and Hasse [4] in terms of gauss sums. Let $a, b \in \mathbf{Q}$ such that neither a nor b nor $a + b$ lie in $\mathbf{Z}$. Suppose further that $(q-1)a$, $(q-1)b$ both lie in $\mathbf{Z}$. Then

$$\frac{g_f((q-1)a)g_f((q-1)b)}{g_f((q-1)(a+b))} = -\sum_{x+y=1} (\operatorname{Teich} x)^{-(q-1)a}(\operatorname{Teich} y)^{-(q-1)b}, \tag{7.2}$$

the sum being over all $x, y \in \mathbf{F}_q^*$ such that $x + y = 1$. (Thus $x = 0, 1$ are excluded.) The left side of (7.2) may be written in terms of Γ_p values by means of (5.13).

Our object is to give a more immediate cohomological interpretation of the Γ_p-analogue of the left side of (7.1). We use the ideas of §6 above extended to the case of two variables. If a_1, a_2 are p-integral rational numbers let

$$a_i' = \langle pa_i \rangle, \qquad t_i = pa_i - a_i', \qquad i = 1, 2,$$

and let

$$\omega_{a_1,a_2} = X_1^{a_1}X_2^{a_2} \exp \pi(X_1 + X_2)\frac{dX_1 \wedge dX_2}{X_1X_2} = \omega_{a_1}(X_1) \wedge \omega_{a_2}(X_2).$$

Then by (6.7)

$$\phi\omega_{a_1,a_2} = \frac{1}{\Gamma_p(a_1')\Gamma_p(a_2')\pi^{t_1+t_2}}\, \omega_{a_1',a_2'} \quad \text{mod exact}. \tag{7.3}$$

We now follow the classical proofs, writing ω_{a_1,a_2} in terms of new variables u, λ_1, λ_2:

$$X_1 = u\lambda_1, \qquad X_2 = u\lambda_2, \qquad 1 = \lambda_1 + \lambda_2.$$

We define

$$\tau_{a_1,a_2} = \lambda_1^{a_1}\lambda_2^{a_2}\left(\frac{d\lambda_1}{\lambda_1} - \frac{d\lambda_2}{\lambda_2}\right)$$

and obtain

$$\omega_{a_1,a_2}(X) = -\omega_{a_1+a_2}(u) \wedge \tau_{a_1,a_2}(\lambda). \tag{7.4}$$

We use (7.3) and (7.4) to calculate $\phi(\tau_{a_1,a_2})$, the idea being to write the left side of (7.3) as

$$\phi(\omega_{a_1,a_2}) = -\phi(\omega_{a_1+a_2}(u)) \wedge \phi(\tau_{a_1,a_2}).$$

The first factor $-\phi(\omega_{a_1+a_2})$ may be computed from (6.7); i.e. let $\bar{c}$ be the minimal representative mod $\mathbf{Z}$ of $c = a_1 + a_2$ so by (6.3)

$$\omega_c = \omega_{\bar{c}} \frac{\Gamma(c)}{\Gamma(\bar{c})}(-\pi)^{(\bar{c}-c)}$$

and

$$\phi(\omega_{\bar{c}}) = \frac{1}{\Gamma_p(c')\pi^{t_3}}\,\omega_{c'}$$

where $c' = \langle p\bar{c}\rangle = \langle pc\rangle, t_3 = p\bar{c} - c'$. Combining these steps gives

$$\phi(\omega_{a_1,a_2}) = K_1 \cdot \omega_{c'}(u) \wedge (\tau_{a_1,a_2}(\lambda))^{\phi} \tag{7.5}$$

where

$$K_1 = -\frac{\Gamma(c)}{\Gamma(\bar{c})}\frac{(-\pi)^{\bar{c}-c-t_3}}{\Gamma_p(c')}(-1)^{t_3}.$$

We now calculate the right side of (7.3) by means of (7.4). This gives

$$\text{right side of (7.3)} = K_2\omega_{a_1'+a_2'}(u) \wedge \tau_{a_1',a_2'}(\lambda) \tag{7.6}$$

where

$$K_2 = -1/\left(\Gamma_p(a_1')\Gamma_p(a_2')\pi^{t_1+t_2}\right).$$

Again by (6.3)

$$\omega_{a_1'+a_2'}/\Gamma(a_1' + a_2') = \omega_{c'}(-\pi)^{c'-a_1'-a_2'}/\Gamma(c'). \tag{7.7}$$

We now equate the right sides of (7.5) and (7.6) and use equation (7.7) to obtain

$$(\tau_{a_1,a_2}(\lambda))^{\phi} \equiv \frac{K(a_1, a_2)}{B_p(a_1', a_2')}\,\tau_{a_1',a_2'}(\lambda) \quad \text{mod exact} \tag{7.8}$$

where $B_p(a, b) = \Gamma_p(a)\Gamma_p(b)/\Gamma_p(a + b)$ and

$$K(a_1, a_2) = h(a_1, a_2)k(a_1', a_2'),$$

$$h(a_1, a_2) = \frac{\Gamma(\bar{c})}{\Gamma(c)}\,p^{\bar{c}-(a_1+a_2)},$$

$$k(a_1', a_2') = (-1)^{a_1'+a_2'-c'}\frac{\Gamma(a_1' + a_2')}{\Gamma(c')}\frac{\Gamma_p(c')}{\Gamma_p(a_1' + a_2')}.$$

Since $\bar{c} = \langle a_1 + a_2\rangle$, $a_1, a_2, a_1', a_2' \in (0, 1)$, $c' = \langle a_1' + a_2'\rangle$. There are four possibilities as indicated below.

$$h(a_1, a_2) = 1, \qquad 0 < a_1 + a_2 < 1,$$
$$= \frac{1}{p(a_1 + a_2 - 1)}, \qquad 1 < a_1 + a_2 < 2,$$

$$k(a_1', a_2') = a_1' + a_2' - 1, \qquad \begin{cases} a_1' + a_2' \equiv 1 \mod p, \\ 1 < a_1' + a_2' < 2, \end{cases}$$
$$= 1, \qquad \text{otherwise.}$$

Equation (7.8) gives the cohomological interpretation of the p-adic beta function B_p. The formula for the right side of (7.2) in terms of B_p values may be deduced from the left side of (7.2) together with (5.13). It may also be deduced directly from (7.8) by the methods of Washnitzer and Monsky.

8. Gauss's multiplication formula. (Gross-Koblitz). A proof of Gauss's formula for $\Gamma(nX)/\prod_{i=0}^{n-1}\Gamma(X+i)$ by means of integration has been given by Dirichlet [**0**]. For this and other reasons a cohomological proof of the corresponding formula for Γ_p could be anticipated. However, it seems easier to follow Sonine's proof [**6**] of the classical equation which uses difference equations. In brief let $n \in \mathbf{N}, p \nmid n$:

$$\phi_n(X) = \prod_{i=0}^{n-1} \Gamma_p\left(\frac{X+i}{n}\right)/\Gamma_p(X). \tag{8.1}$$

By the functional equation of $\Gamma_p(X)$, we have

$$\frac{\Gamma_p(X+1)}{\Gamma_p(X)} = h_X(n)\frac{\Gamma_p(X/n+1)}{\Gamma_p(X/n)}, \qquad X \in \mathbf{N}, \tag{8.2}$$

where

$$h_X(n) = 1 \quad \text{if } p \mid X,$$
$$= n \quad \text{if } p \nmid X.$$

We deduce $\phi_n(X+1) = (1/h_X(n))\phi_n(X)$ and hence, for $X \in \mathbf{N}$,

$$\phi_n(X) = \phi_n(0)\prod_{i=0}^{X-1}\frac{1}{h_i(n)}$$

and so

$$\frac{\phi_n(X)}{\phi_n(0)} = \left(\frac{1}{n}\right)^{X-1-[(X-1)/p]}. \tag{8.3}$$

The equation is valid for all $X \in \mathbf{N}$ and hence may be extended by continuity to $\mathbf{Z}_p$. Indeed, we may extend $X \mapsto [X/p]$ to $\mathbf{Z}_p$ by writing $X = X_0 + X_1 p + \dots$, $0 < X_0 < p$, and setting $[X/p] = X_1 + pX_2 + \dots$. The expression $(1/n)^{X-[X/p]}$ is then well defined as being $(1/n)^{X_0}(1/n)^{(p-1)(X_1+pX_2+\cdots)}$. The important point is that it is a continuous function of $X \in \mathbf{Z}_p$.

To complete the formula we calculate $\phi_n(0)$ by means of (1.3) to be

$$\phi_n(0) = (-1)^A\Gamma_p\left(\tfrac{1}{2}\right)^B \tag{8.4}$$

where

$$A = \sum_{1<j<n/2} l(j/n),$$

$$B = 0, \quad n \text{ odd},$$
$$= 1, \quad n \text{ even}.$$

We note that $\Gamma_p(\frac{1}{2})^2 = (-1)^{(p+1)/2}$ and so

$$\Gamma_p\left(\tfrac{1}{2}\right) = \pm 1, \qquad p \equiv -1 \bmod 4,$$
$$= \pm\sqrt{-1}, \qquad \equiv +1 \bmod 4.$$

$\Gamma_p(\frac{1}{2})$ is fixed by this and the congruence $\Gamma_p(\frac{1}{2}) \equiv ((p-1)/2)! \mod p$.

The Gauss multiplication formula for Γ_p is given by (8.1), (8.3) and (8.4).

REMARKS. (1) The results of §8 should be compared via (5.13) with the Hasse-Davenport relation [4, equation (0.9)].

For $b \in \mathbf{Q}$, $b(q-1) \in \mathbf{N}$, $d|(q-1)$ we have

$$\prod_{j=0}^{d-1} g_f\left((q-1)\left(\frac{j}{d}+b\right)\right) = cg_f((q-1)db) \cdot \prod_{j=0}^{d-1} g_f\left(\frac{j}{d}(q-1)\right)$$

where $c = (\text{Teich } d)^{(q-1)db}$.

(2) Our knowledge of Γ_p should be tested against other reported relations involving gauss sums.

(i) For $b \in \mathbf{Q}$, $b(q-1) \in \mathbf{N}$, $n \in \mathbf{N}$:

$$g_{fn}(b(q^n-1)) = \left(g_f(b(q-1))\right)^n \qquad [4, (0.8)].$$

(ii) Let l be prime, let $n =$ order of q in $\mathbf{F}_l^*$ so $\{1, q, \ldots, q^{n-1}\}$ is a subgroup of $\mathbf{F}_l^*$ of index $e = (l-1)/n$. Let $i_1, \ldots, i_e$ be integers which represent the factor group. Let $a \in \mathbf{Q}$, $a(q-1) \in \mathbf{N}$; then

$$cg_f(la(q-1)) \cdot \prod_{j=1}^{e} g_{fn}\left(i_j \frac{q^n-1}{l}\right)$$
$$= g_f(a(q-1)) \cdot \prod_{j=1}^{e} g_{fn}\left(\left(a+\frac{i_j}{l}\right)(q^n-1)\right)$$

where $c = (\text{Teich } l)^{la(q-1)}$ (Langlands, unpublished).

(iii) Let l be a prime, $l|(q-1)$. Let $a \in \mathbf{Q}$, $a(q-1) \in \mathbf{N}$, $a((q-1)/l) \notin \mathbf{N}$. Then

$$cg_f(a(q-1)) \cdot \prod_{i=1}^{l-1} g_f\left(\frac{q-1}{l} i\right) = g_{fl}\left(a\frac{q^l-1}{l}\right)$$
$$= g_{fl}\left((a+j)\frac{q^l-1}{l}\right) \qquad \forall j \in \mathbf{N}$$

where again $c = (\text{Teich } l)^l a(q-1)$ (Dwork, also Langlands, unpublished).

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